Robotics:
Inverse kinematics

Sami Haddadin

Institute of Robotics and Mechatronics
German Aerospace Center (DLR), Germany

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Problem partition

Split the problem of placing the tool frame $\{T\}$ with respect to the station frame $\{S\}$ into two problems

1. Frame transformations are performed to find the wrist frame $\{W\}$ relative to the base frame $\{B\}$
2. Inverse kinematics are use to solve for the joint angles $\boldsymbol{\theta}$

Solvability

The problem of solving kinematic equations of a manipulator is nonlinear:

$$^{0}T_n(\boldsymbol{\theta}) \rightarrow \boldsymbol{\theta} \tag{1}$$

Example *Puma 560*: Given 16 numerical values for $^{0}T_6$ (four of which are trivial), solve for the six joint angles $\theta_1 \dots \theta_6$.

Solvability

- 12 equations and 6 unknowns
- among nine equations from ${}^{0}R_6$ only three equations are independent
- $\rightarrow$ six equations with six unknown
- nonlinear transcendental[1] equations
- for many $\alpha_i, a_i \neq 0$ it gets quite complex
- nonlinear equations necessitate to discuss **existence of solutions, multiple solutions, and the method of solution**

[1] equations, which can only be formulated implicitly and are in most cases only geometrically or numerically solvable, but not analytically.

Existence of solutions

For a solution to exist it must lie within the workspace.

Dexterous workspace $\mathcal{D}$:

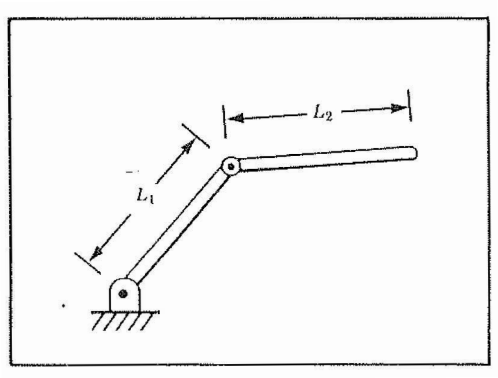
- that volume of space which the robot end-effecto can reach with all orientations
- i.e., at each point in $\mathcal{D}$, the end-effecto can be arbitrarily oriented

Reachable workspace $\mathcal{R}$:

- that volume of space, which the robot can reach in at least one orientation

$\rightarrow \mathcal{D} \subseteq \mathcal{R}$

Example: two-link planar robot



Example: two-link planar robot

$l_1 = l_2$:

- $\mathcal{R}$ consists of a disc with radius $2l_1$
- $\mathcal{D}$ consists of a single point: the origin

$l_1 \neq l_2$:

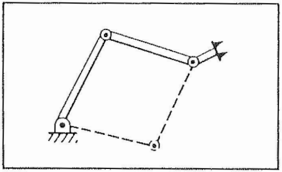
- $\mathcal{R}$ consists of a ring with $r_{\min} = |l_1 - l_2|$ and $r_{\max} = l_1 + l_2$
- $\mathcal{D}$ none

Inside $\mathcal{R}$ there are 2 possible orientations of the end-effector, while on the oundary there is only one possible configuration. (Please note, this assumes at least $\forall i : 0^o \leq \theta_i \leq 360^o$)

Existence of solutions

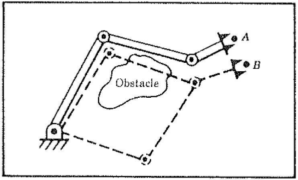
- a robot with $n < 6$ cannot attain a general goal in position and orientation in 3-space
- planar manipulator cannot reach out of z-plane
- in general, the workspace of a robot with $n < 6$ is a subset of a **subspace**, which can be associated with any type of robot
- typical problem: If I cannot reach my goal, what is the closest solution?
- please note: the operator usually refers to $\{T\}$, given $\{G\}$. However, we usually do not refer to $\{T\}$, rather to $\{W\}$

Multiple solutions



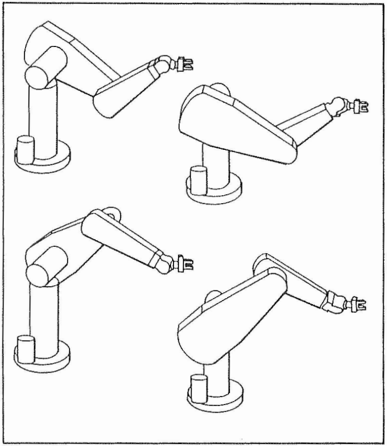
- planar arm with $n = 3$ revolute joints has a large dexterous workspace in the plane
- any position in the interior of its workspace can be reached with any orientation

Multiple solutions



- problem: which solution?
- what is closeness? one possibility: (weighted) distance in joint space
- take into account obstacles

Example: Puma 560



Mutltiple solutions

- Four solutions shown for the same hand configuration
- Four more solutions are given by

$$\begin{aligned} \theta_4' &= \theta_4 + 180^o \\ \theta_5' &= -\theta_5 \\ \theta_6' &= \theta_6 + 180^o \end{aligned}$$

$\rightarrow$ eight solutions for one single goal. For a general manipulator with 6DoF, there are up to 16 solutions.

Definition: solvability

A manipulator will be considered **solvable** if the joint variables can be determined by an algorithm which allows one to determine all the joint variables associated with a given position and orientation.

Method of solution

- unlike linear equations there are no general algorithms which may be employed to solve a set of nonlinear equations
- two classes:
    1. closed form solutions
    2. numerical solutions
- Numerical solutions: potentially problematic due to local minima and execution time
- two classes of closed form solutions
    1. algebraic
    2. geometric

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Method of solution: numerical methods

the forward kinematics may be interpreted as

$$\mathbf{w} = f(\boldsymbol{\theta}), \tag{2}$$

with $\mathbf{w} \in \mathbb{R}^n$. Rearranging (2):

$$f(\boldsymbol{\theta}) - \mathbf{w} = \mathbf{0} \tag{3}$$

This can e.g. be solved by the **Newton-Raphson Method**:

$$\boldsymbol{\theta}^{(\nu+1)} := \boldsymbol{\theta}^{(\nu)} - J^{-1}(\boldsymbol{\theta}^{(\nu)})(f(\boldsymbol{\theta}^{(\nu)}) - \mathbf{w}), \tag{4}$$

with $\nu = 0, 1, ...$ is the iteration index and

$$J = [\partial f_i / \partial \theta_j] \in \mathbb{R}^{n \times n} \tag{5}$$

is the Jacobi matrix (Jacobian). The iteration is stopped for ν_{max} or for $||\boldsymbol{\theta}^{(\nu+1)} - \boldsymbol{\theta}^{(\nu)}|| < \epsilon$

Method of solution: numerical methods

- cost intensive and slow inverse algorithm
- area of attraction of the iteration is for ambiguity
- be aware of problems for **singularity**: $\det(J) = 0$

Method of solution

- important result: all systems with revolute and prismatic joints having a total degree six DoF in a single series chain are solvable.

However, this general solution is a numerical one.

Closed form solutions

- closed form solutions exist for some robots with intersecting joint axes and/or many $\alpha_i = 0$
- virtually all industrial manipulators are designed sufficiently simple so that a closed form solution exists
- **sufficient condition:** robot with six revolute joints has closed solution if three neighboring axes intersect
- Puma 560: axes $4, 5, 6$ intersect

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Algebraic solution

Consider the three-link planar manipulator

$$
{}^{B}T_{W} = \begin{bmatrix} c_{123} & -s_{123} & 0 & l_1 c_1 + l_2 c_{12} \\ s_{123} & c_{123} & 0 & l_1 s_1 + l_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \tag{6}
$$

i	α_i	a_{i-1}	d_i	θ_1
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3

Algebraic solution

planar robot $\rightarrow x, z, \Phi$ are given

$$ {}^{B}T_{W} = \begin{bmatrix} c_\Phi & -s_\Phi & 0 & \mathrm{x} \\ s_\Phi & c_\Phi & 0 & \mathrm{y} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \tag{7} $$

Set of nonlinear equations:

$$ c_\Phi = c_{123} \tag{8} $$

$$ s_\Phi = s_{123} \tag{9} $$

$$ x = l_1 c_1 + l_2 c_{12} \tag{10} $$

$$ x = l_1 s_1 + l_2 s_{12} \tag{11} $$

Algebraic solution

solution:

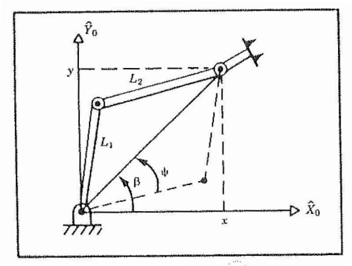
$$\begin{aligned}\theta_1 &= \mathrm{atan2}(y, x) - \mathrm{atan2}(k_2, k_1) \\ \theta_2 &= \mathrm{atan2}(s_2, c_2) \\ \theta_3 &= \Phi - \theta_1 - \theta_2 \end{aligned} \quad (12)$$

Calculations of algebraic solution: **blackboard**

Geometric solution

Try to decompose the spatial geometry of the arm into several plane geometry problems.

- can be done easily for many manipulators especially if $\alpha_i = 0$ or $\pm 90^o$
- joint angles can be solved by using the plane of geometry



Calculations of geometric solution: **blackboard**

Pieper's solution

Pieper treated manipulators with 6DoF in which three consecutive axes intersect.

- Craig: case of all six joints revolute
- last three axes intersect
- applies to other configurations which include prismatic joints as well
- applies to most commercially available industrial robots

Repeatability & accuracy

Accuracy is a term often confused with resolution and repeatability. Three factors are brought together to describe the characteristic or specification known as accuracy as related to robots:

accuracy:

- the resolution of the control components
- the inaccuracies of the mechanical components
- an arbitrary, never-before-approached fixed position (target)

$\rightarrow$ only meaningful if the robot is not only used as a tape recorder

Repeatability & accuracy

repeatability:

- the ability of the robot to reposition itself to a position which it was previously commanded or trained.

Repeatability and accuracy are similar: however, they define slightly different performance concepts