On computing nullspace bases – a fault detection perspective

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Outline

- fault detection problem
- nullspace method
- minimal polynomial bases
- simple rational bases
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- nullspace operations and updating
- conclusions

Fault detection problem (FDP)

Input-output model with additive faults:

$$\mathbf{y}(s) = G_u(s)\mathbf{u}(s) + G_f(s)\mathbf{f}(s) + G_d(s)\mathbf{d}(s),$$

where $y(t) \in \mathbb{R}^p$ – system output, $u(t) \in \mathbb{R}^{m_u}$ – system control input, $f(t) \in \mathbb{R}^{m_f}$ – system fault input, and $d(t) \in \mathbb{R}^{m_d}$ – system disturbance input.

FDP: Determine a least order stable and proper fault detection filter

$$\mathbf{r}(s) = Q(s) \begin{bmatrix} \mathbf{y}(s) \\ \mathbf{u}(s) \end{bmatrix}$$

such that $(1) \quad r(t) = 0$ when $f(t) = 0$
$(2) \quad r(t) \neq 0$ when $f(t) \neq 0$

Algebraic design and solvability conditions

FDP: Determine stable and proper $Q(s)$ such that

$$(1) \quad \Rightarrow \quad Q(s)\left[\begin{array}{cc} G_u(s) & G_d(s) \\ I_{m_u} & 0 \end{array}\right] = 0$$

$$(2) \quad \Rightarrow \quad R_{f_i}(s) := Q(s)\left[\begin{array}{c} G_{f_i}(s) \\ 0 \end{array}\right] \neq 0, \;\; i = 1, \ldots, m_f$$

where $G_{f_i}(s)$ is the i-th column of $G_f(s)$.

Solvability: The FDP is solvable iff

$$\operatorname{rank}\,[\,G_d(s)\; G_{f_i}(s)\,] > \operatorname{rank} G_d(s), \quad i = 1, \ldots, m_f$$

Nullspace method (Frisk & Nyberg, 2001; V, 2003)

1. Compute a minimal left nullspace basis $N_l(s)$ s.t. $N_l(s)G(s) = 0$, where

$$G(s) = \begin{bmatrix} G_u(s) & G_d(s) \\ I_{m_u} & 0 \end{bmatrix}$$

2. Check solvability: $N_l(s)\begin{bmatrix} G_{f_i}(s) \\ 0 \end{bmatrix} \neq 0,\ i=1,\ldots,m_f$;
   STOP if no solution exists.
3. Determine a rational vector $h(s)$ such that

$$Q(s) := h(s)N_l(s)$$

   is stable, has the least McMillan degree and $R_{f_i}(s) \neq 0,\ i = 1,\ldots,m_f$.

Computational problems to be addressed

Main quantities to be computed:

(a) $Q(s)$ – fault detection filter ($Q(s) = N_l(s)$ good first candidate)

(b) $R_f(s) := Q(s)\begin{bmatrix} G_f(s) \\ 0 \end{bmatrix}$ – fault-to-residual transfer matrix

Updating of main quantities: (e.g., for stable and least order design)

(a) $Q(s) \leftarrow h(s)Q(s)$

(b) $R_f(s) \leftarrow h(s)R_f(s)$

Relevant aspects for algorithm selection:

– plant and detector models are usually in state-space form

– numerical reliability of computations necessary

– flexibility to address the least order synthesis aspect

Minimal polynomial nullspace bases

Polynomial basis: For a $p \times m$ $G(s)$, with $r = \operatorname{rank} G(s) < p$, let $N_l(s)$ be a $(p-r) \times p$ polynomial basis of the left nullspace of $G(s)$.

Row degrees (or indices): n_i – greatest degree of the i-th row of $N_l(s)$.

Order of $N_l(s)$: $n_l = \sum_{i=1}^{p-r} n_i$

Minimal polynomial basis: Has least order among all polynomial bases.
$\Rightarrow n_i$, $i = 1, \dots, n-p$ are called *minimal row indices*.

Useful property: Assume $n_1 \leq n_2 \leq \dots \leq n_{p-r}$ and let $h_i \neq 0$.
If $h = [\, h_1, \dots, h_i, 0, \dots 0 \,]$, then $hN_l(s)$ has degree n_i.

Properties of minimal basis

Theorem (Forney, 1975; Kailath, 1980): Let $N_l(\lambda)$ be a minimal polynomial basis with row indices n_i, $i = 1, \ldots, p - r$. Then the following holds:

1. The row indices are *unique* up to row permutations.
2. $N_l(\lambda)$ is *irreducible* (i.e., has full row rank $\forall \lambda \in \mathbb{C}$)
   $\Leftrightarrow N_l(\lambda)$ has no finite zeros.
3. $N_l(\lambda)$ is *row reduced* (i.e., the *leading row coefficient matrix* formed from the coefficients of the highest row degrees, has full row rank)
   $\Leftrightarrow N_l(\lambda)$ has no infinite zeros.

Polynomial nullspace based methods

Nullspace computation methods:

– resultant methods for polynomial matrices (Antoniou *et al.*, 2005)
– matrix pencil methods for state-space models (Beelen, 1987)

Pros: – conceptually simple approach
(applicable to small order systems)
– building of a least order detector straightforward
(constant linear combination of basis vectors)

Cons: – computation of nullspace may not be reliable for resultant methods
(especially for high order systems)
– checking solvability conditions requires special computations
– computational detours may be necessary (polynomial $\rightleftarrows$ state-space)

Simple rational nullspace bases

Polynomial basis: Let $N_l(s)$ be a $(p-r)\times p$ minimal polynomial basis of the left nullspace of a $p\times m$ $G(s)$, where $r = \mathrm{rank}G(s) < p$ with minimal row indices n_i, $i = 1,\ldots,p-r$. Define

$$M(s) = \mathrm{diag}\left(\frac{1}{d_1(s)},\cdots,\frac{1}{d_{p-r}(s)}\right),$$

where $d_i(s)$ is a polynomial of degree n_i (with arbitrary zeros).

Simple proper rational basis: $\widetilde{N}_l(s) := M(s)N_l(s)$

Main properties of $\widetilde{N}_l(s)$:
- McMillan degree of $\widetilde{N}_l(s)$ is $n_l = \sum_{i=1}^{p-r} n_i$
- $D_l := \lim_{s\rightarrow\infty} \widetilde{N}_l(s)$ has full row rank $p-r$

Useful property: Assume $n_1 \leq n_2 \leq \ldots \leq n_{p-r}$, $d_i(s)|d_j(s)$ for $i<j$, and $h_i \neq 0$. If $h = [\,h_1,\ldots,h_i,0,\ldots 0\,]$, then $h\widetilde{N}_l(s)$ has McMillan degree n_i.

Rational nullspace based methods

Nullspace computation methods:
– minimal basis using matrix pencil methods (Beelen, 1987, V., 2003)
– simple basis from a minimal basis (this paper)

Pros: – state space computations based numerically reliable approach
– building a least order detector straightforward
– checking solvability conditions requires no special computations
– no model conversions necessary
– explicit updating formulas can be employed
– numerical software available in the Descriptor Toolbox (V, 1999-2008)

Cons: None

State space computations

State space realization of $p \times m$ $G(s)$**:**

$$
\begin{aligned}
E\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{aligned}
$$

such that

$$G(s) = C(sE - A)^{-1}B + D$$

Computation of nullspace bases

System matrix:

$$S(s) = \begin{bmatrix} A - sE & B \\ C & D \end{bmatrix}$$

Fact: If $Y_l(s)$ is a left nullspace basis of $S(s)$, then

$$N_l(s) = Y_l(s) \begin{bmatrix} 0 \\ I_p \end{bmatrix}$$

is a left nullspace basis of $G(s)$.

Computation of proper rational nullspace bases

Kronecker-like form: Let Q and Z be orthogonal matrices s.t.

$$QS(s)Z = \left[\begin{array}{cc} A_r - sE_r & A_{r,l} - sE_{r,l} \\ \hline 0 & A_l - sE_l \\ \hline 0 & C_l \end{array} \right]$$

where $(A_l - sE_l, C_l)$ is observable, E_l is non-singular, and $A_r - sE_r$ has full normal row rank. Then

$$Y_l(s) = \left[\begin{array}{c|c|c} 0 & C_l(sE_l - A_l)^{-1} & I \end{array} \right] Q$$

$$N_l(s) = Y_l(s) \left[\begin{array}{c} 0 \\ I_p \end{array} \right] = C_l(sE_l - A_l)^{-1}B_l + D_l$$

$\Rightarrow$ The algorithm to compute $N_l(s)$ is numerically backward stable.

Some properties of $N_l(s)$

P1. If the realization of $G(s)$ is minimal then $N_l(s)$ is a minimal proper left nullspace basis of least McMillan degree (order n_l).

P2. If the realization of $G(s)$ is minimal then $N_l(s)$ maximally controllable $\Leftrightarrow (A_l + KC_l - sE_l, B_l + KD_l)$ controllable $\forall K$.

Remark. $N_l(s)$ is in general not stable and not simple.

Computation of stable proper nullspace bases

Consider the left transformation

$$U = \left[\begin{array}{ccc} I & 0 & 0 \\ 0 & I_{n_l} & K \\ 0 & 0 & I_{p_l} \end{array}\right]$$

$$UQS(s)Z = \left[\begin{array}{cc} A_r - sE_r & A_{r,l} - sE_{r,l} \\ \hline 0 & A_l + KC_l - sE_l \\ \hline 0 & C_l \end{array}\right]$$

$\Rightarrow$ a stable proper nullspace basis can be determined as

$$\widetilde{N}_l(s) = C_l(sE_l - A_l - KC_l)^{-1}(B_l + KD_l) + D_l$$

with K a stabilizing output injection matrix.

Computation of simple bases

Minimal proper rational nullspace basis: $N_l(s) = C_l(sE_l - A_l)^{-1}B_l + D_l$

Theorem: For each $i = 1, \ldots, p-r$, choose vector $h_i = e_i^T$ and let K_i be such that

$$v_i(s) := h_i C_l(sE_l - A_l - K_i C_l)^{-1}(B_l + K_i D_l) + h_i D_l$$

has the least possible McMillan degree. Then

$$V(s) = \begin{bmatrix} v_1(s) \\ v_2(s) \\ \vdots \\ v_{p-r}(s) \end{bmatrix}$$

is a simple proper minimal rational basis.

Important property

Staircase structure:

$$\begin{bmatrix} A_l - sE_l \\ C_l \end{bmatrix} = \begin{array}{cl} \left[\begin{array}{c|c|c|c} A_{\ell,\ell+1} & A_{\ell,\ell} - sE_{\ell,\ell} & \cdots & A_{\ell,1} - sE_{\ell,1} \\ \hline & A_{\ell-1,\ell} & \ddots & \vdots \\ \hline & & \ddots & A_{1,1} - sE_{1,1} \\ \hline & & & A_{0,1} \end{array}\right] & \begin{array}{l} \mu_\ell \\ \mu_{\ell-1} \\ \mu_1 \\ \mu_0 = p - r \end{array} \\ \begin{array}{cccc} \mu_{\ell+1} = 0 & \qquad \mu_\ell \qquad & \qquad \cdots \qquad & \mu_1 \end{array} & \end{array}$$

$A_{i,i+1}$ full column rank, upper triangular

Degrees of basis vectors: There are $\mu_{i-1} - \mu_i$ vectors of degree $i-1$
$\Rightarrow$ There are $\mu_0 - \mu_i$ vectors up to degree $i-1$

Linear combination of basis vectors up to a given degree $i-1$:
Choose $h_i = [\,0 \ \cdots \ 0 \ * \ \cdots \ *\,]$ with $\mu_0 - \mu_i$ non-zero trailing elements.

Basic problem

Problem: For given h determine K such that

$$v(s) = hC_l(sE_l - A_l - KC_l)^{-1}(B_l + KD_l) + hD_l$$

has least McMillan degree.

$\Leftrightarrow$ Determine K such that $(A_l + KC_l - sE_l, hC_l)$ is maximally unobservable because $(A_l + KC_l - sE_l, B_l + KD_l)$ is controllable $\forall\ K$

Approach: Determine K by solving a minimal dynamic cover problem using numerically reliable algorithms (V, 2003; 2004).

Basic computation

Minimal cover algorithm output: Matrices K, U and V s.t.

$$
\begin{aligned}
U(A_l + KC_l - sE_l)V &= \left[\begin{array}{c|c} \widehat{A}_{11} - s\widehat{E}_{11} & * \\ \hline 0 & \widehat{A}_{22} - s\widehat{E}_{22} \end{array}\right] \\
U(B_l + KD_l) &= \left[\begin{array}{c} \widehat{B}_1 \\ \hline \widehat{B}_2 \end{array}\right] \\
hC_lV &= \left[\begin{array}{c|c} 0 & \widehat{c}_{22} \end{array}\right]
\end{aligned}
$$

Least order solution: $v(s) = \widehat{c}_{22}(s\widehat{E}_{22} - \widehat{A}_{22})^{-1}\widehat{B}_2 + hD_l$

Operations with nullspace bases

$$[\,G(s)\ F(s)\,] = \left[\begin{array}{c|cc} A - sE & B & B_f \\ \hline C & D & D_f \end{array}\right]$$

1. Compute $N_l(s)F(s)$, given a minimal proper left nullspace basis of $G(s)$

$$N_l(s) = C_l(sE_l - A_l)^{-1}B_l + D_l$$

   FD Application: checking solvability conditions

2. Compute $\widetilde{N}_l(s)F(s)$, given a minimal proper left nullspace basis of $G(s)$

$$\widetilde{N}_l(s) = C_l(sE_l - A_l - KC_l)^{-1}(B_l + KD_l) + D_l$$

   FD Application: checking solvability conditions with a stable basis

3. Compute $v(s)F(s)$, given least order $v(s) = h\widetilde{N}_l(s)$.
   FD Application: checking solvability conditions for least order designs

Explicit realization for $N_l(s)F(s)$

$$S_e(s) := \left[\begin{array}{cc|c|c} A - sE & B & 0 & B_f \\ C & D & I & D_f \end{array}\right] := [\, S(s) \,|\, S_l \,|\, S_f \,]$$

Let Q and Z be orthogonal matrices to reduce $S(s)$ to Kronecker-like form:

$$QS_e(s)\mathrm{diag}(Z, I) = \left[\begin{array}{cccc} A_r - sE_r & A_{r,l} - sE_{r,l} & * & * \\ \hline 0 & A_l - sE_l & B_l & \widetilde{B}_f \\ \hline 0 & C_l & D_l & \widetilde{D}_f \end{array}\right]$$

$$\Rightarrow \quad [\, N_l(s) \quad N_l(s)F(s) \,] = \left[\begin{array}{c|cc} A_l - sE_l & B_l & \widetilde{B}_f \\ \hline C_l & D_l & \widetilde{D}_f \end{array}\right]$$

The algorithm to compute $N_l(s)F(s)$ is numerically backward stable.

Explicit realization for $\widetilde{N}_l(s)F(s)$

$$U = \begin{bmatrix} I & 0 & 0 \\ 0 & I & K \\ 0 & 0 & I \end{bmatrix}$$

$$UQS_e(s)\mathrm{diag}(Z, I) = \left[\begin{array}{cccc} A_r - sE_r & A_{r,l} - sE_{r,l} & * & * \\ \hline 0 & A_l + KC_l - sE_l & B_l + KD_l & \widetilde{B}_f + K\widetilde{B}_f \\ \hline 0 & C_l & D_l & \widetilde{D}_f \end{array}\right]$$

$$\Rightarrow \quad [\, \widetilde{N}_l(s) \quad \widetilde{N}_l(s)F(s) \,] = \left[\begin{array}{c|cc} A_l + KC_l - sE_l & B_l + KD_l & \widetilde{B}_f + K\widetilde{D}_f \\ \hline C_l & D_l & \widetilde{D}_f \end{array}\right]$$

Explicit realization of least order $h\widetilde{N}_l(s)F(s)$

$$
\begin{aligned}
U(A_l + KC_l - sE_l)V &= \left[\begin{array}{c|c} \widehat{A}_{11} - s\widehat{E}_{11} & * \\ \hline 0 & \widehat{A}_{22} - s\widehat{E}_{22} \end{array}\right] \\
U(B_l + KD_l) &= \left[\begin{array}{c} \widehat{B}_1 \\ \hline \widehat{B}_2 \end{array}\right], \quad U(\widetilde{B}_f + K\widetilde{D}_f) = \left[\begin{array}{c} \widehat{B}_{f,1} \\ \hline \widehat{B}_{f,2} \end{array}\right] \\
hC_lV &= \left[\begin{array}{c|c} 0 & \widehat{c}_{22} \end{array}\right]
\end{aligned}
$$

$$
\Rightarrow \quad [\, h\widetilde{N}_l(s) \quad h\widetilde{N}_l(s)F(s)\,] = \left[\begin{array}{c|cc} \widehat{A}_{22} - s\widehat{E}_{22} & \widehat{B}_2 & \widehat{B}_{f,2} \\ \hline \widehat{c}_{22} & hD_l & h\widetilde{D}_f \end{array}\right]
$$

Conclusions

- The rational nullspace based approach allows a numerically satisfactory solution of all computational problems involved in the design of residual generators: nullspace computation, checking solvability conditions, filter stabilization/dynamics assignment, least order filter design
- Updating techniques based efficient and numerically reliable computation of the achievable complete fault signature (V, Safeprocess'09, paper in preparation)
- Updating technique based efficient solution of the Fault Detection and Isolation Problem using a bank of scalar output detectors (V, 2007)
- The new algorithms are implemented as flexible and robust numerical software tools in the Fault Detection Toolbox of DLR (based on the Descriptor Systems Toolbox – V, 1999–2008)