Numerical Methods for Continuous-time Periodic Systems

A Multiple-Shooting Algorithmic Paradigm

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Outline

- periodic systems preliminaries
- periodic Riccati differential equations
- *single shooting* solution methods
- *multiple shooting* solution methods
- applications to various computational problems
- some open computational problems

Continuos-time periodic systems

System description:

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t)\\ y(t) &= C(t)x(t) + D(t)u(t)\end{aligned}$$

where $A(t)$, $B(t)$, $C(t)$, $D(t)$ are T-periodic matrices.

Some periodic control applications:

– satellite attitude periodic control on low earth orbits
– helicopter forward flight periodic control for vibration reduction
– wind turbine periodic control for reducing blade loads
– path following (or orbital stabilization) for underactuated robots

Preliminaries

Transition matrix: For a given $A(t) \in R^{n\times n}$, $\Phi_A(t,\tau)$ satisfies

$$\frac{\partial \Phi_A(t,\tau)}{\partial t} = A(t)\Phi_A(t,\tau), \quad \Phi_A(\tau,\tau) = I_n$$

Monodromy matrix: $\Psi_A(\tau) := \Phi_A(\tau + T, \tau)$ for $A(t)$ T-periodic.

Characteristic multipliers: λ_i – eigenvalues of $\Psi_A(\tau)$ (independent of τ).

Characteristic exponents: $\mu_i = \frac{1}{T}\log\lambda_i$

Stability: $A(t)$ is stable $\Leftrightarrow |\lambda_i| < 1 \Leftrightarrow \mathrm{Re}\,\mu_i < 0$.

Similarity transformations

Lyapunov similarity transformations: If $x(t) = P(t)\tilde{x}(t)$ with $P(t)$ T-periodic $\Rightarrow$

$$\begin{aligned} \dot{\tilde{x}}(t) &= \widetilde{A}(t)\tilde{x}(t) + \widetilde{B}(t)u(t) \\ y(t) &= \widetilde{C}(t)\tilde{x}(t) + D(t)u(t) \end{aligned}$$

$$\begin{aligned} \widetilde{A}(t) &= P^{-1}(t)A(t)P(t) - P^{-1}(t)\dot{P}(t) \\ \widetilde{B}(t) &= P^{-1}(t)B(t) \\ \widetilde{C}(t) &= P(t)C(t) \end{aligned}$$

Transformation of transition matrix: $\Phi_{\widetilde{A}}(t,\tau) = P^{-1}(t)\Phi_A(t,\tau)P(\tau)$

Preserved properties:

– characteristic multipliers/exponents $\Rightarrow$ basis of Floquet-analysis

– input-output operator (thus, various system norms)

Periodic Riccati differential equations

Problem: For $A(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix}$ compute $P(t) = \begin{bmatrix} I & 0 \\ X(t) & I \end{bmatrix}$
such that $\widetilde{A}(t) := P^{-1}(t)A(t)P(t) - P^{-1}(t)\dot{P}(t)$

$$\widetilde{A}(t) = \begin{bmatrix} A_{11}(t) + A_{12}(t)X(t) & A_{12}(t) \\ 0 & A_{22}(t) - X(t)A_{12}(t) \end{bmatrix}$$

$$0 = -\dot{X}(t) - X(t)A_{11}(t) + A_{22}(t)X(t) - X(t)A_{12}(t)X(t) + A_{21}(t)$$

Periodic nonsymmetric Riccati differential equation
Periodic symmetric Riccati differential equation:
$A_{11}(t) = -A_{22}^T(t)$, $A_{21}(t) = A_{21}^T(t)$, $A_{12}(t) = A_{12}^T(t)$
Periodic Lyapunov differential equation:
$A_{11}(t) = -A_{22}^T(t)$, $A_{21}(t) = A_{21}^T(t)$, $A_{12}(t) = 0$
Periodic Sylvester differential equation: $A_{12}(t) = 0$

Periodic Riccati differential equations (PRDE)

Problem: For $A(t)$, $Q(t)$ and $R(t)$ T-periodic with $Q(t) = Q^T(t)$ and $R(t) = R^T(t)$, compute the T-periodic stabilizing solution $X(t)$ of the PRDEs

$$\begin{aligned} \dot{X}(t) &= A(t)X(t) + X(t)A^T(t) + R(t) - X(t)Q(t)X(t) \\ -\dot{X}(t) &= A^T(t)X(t) + X(t)A(t) + Q(t) - X(t)R(t)X(t) \end{aligned}$$

Main applications:

- periodic optimal control
- periodic filtering
- $\mathcal{H}_\infty$-norms of periodic systems

Periodic Lyapunov differential equations (PLDE)

Problem: For $A(t)$ and $C(t)$ T-periodic with $C(t) = C^T(t)$, compute the T-periodic solution $X(t)$ of the PLDEs

$$\begin{aligned} \dot{X}(t) &= A(t)X(t) + X(t)A^T(t) + C(t) \\ -\dot{X}(t) &= A^T(t)X(t) + X(t)A(t) + C(t) \end{aligned}$$

Main applications:

- analysis of controllability/observability
- solution of periodic stabilization problems
- computing Hankel- and $\mathcal{H}_2$-norms of periodic systems
- solution of periodic Riccati differential equations using Newton's method

Periodic Sylvester differential equations (PSDE)

Problem: For $A(t)$, B(t) and $C(t)$ T-periodic, compute the T-periodic solution $X(t)$ of the PSDE

$$\dot{X}(t) = A(t)X(t) + X(t)B(t) + C(t)$$

Main application:

- solution of periodic eigenvalue assignment problems
- eigenstructure splitting

Periodic generator method (single shooting)

Basic idea:

1. Compute $X(0)$ such that $X(t)$ satisfies $X(0) = X(T)$.
2. Integrate the differential equation to compute the solution on $[0, T]$.

Computational approach: $\Psi_{\widetilde{A}}(0) = P^{-1}(0)\Psi_A(0)P(0) \Leftrightarrow$

$$\begin{bmatrix} \Psi_{\widetilde{A}}^{11} & \Psi_{\widetilde{A}}^{12} \\ 0 & \Psi_{\widetilde{A}}^{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ -X(0) & I \end{bmatrix} \begin{bmatrix} \Psi_A^{11} & \Psi_A^{12} \\ \Psi_A^{21} & \Psi_A^{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ X(0) & I \end{bmatrix} \quad \Rightarrow$$

$X(0)$ fulfills appropriate algebraic Riccati, Lyapunov or Sylvester equations!

Expected difficulties:

– unstable $A(t)$ leads to unstable integration of ODEs to determine $\Psi_A(0)$
– severe accuracy losses expected for large T

Computational example

$A(t) = \begin{bmatrix} 0 & 1 \\ -2\dot{\alpha}(t) & 6 - 2\alpha(t) \end{bmatrix}$, where $\alpha(t) = 15 + 5\sin t$ has period $T = 2\pi$.

With $P(t) = \begin{bmatrix} 1 & 0 \\ 6 - 2\,\alpha\,(t) & 1 \end{bmatrix} \Rightarrow$

$$\widetilde{A}(t) := P^{-1}(t)A(t)P(t) - P^{-1}(t)\dot{P}(t) = \begin{bmatrix} 6 - 2\alpha(t) & 1 \\ 0 & 0 \end{bmatrix}$$

$\widetilde{A}(t)$ has characteristic exponents: $\mu_1 = 0$ and $\mu_2 = -24$.

Using numerical integration with $reltol = 10^{-10}$ to determine $\Psi(0)$, we get

$\overline{\mu}_1 = -3.9 \cdot 10^{-7}, \quad \overline{\mu}_2 = -2.05... \Rightarrow$ No single accurate digit in μ_2!!!

Alternative computation of transition matrix

Product form of monodromy matrix: Set $\Delta := T/N$, $N > 1$. For $k = 1, \ldots, N$, integrate

$$\dot{F}(t) = A(t)F(t), \quad F(t_0) = I$$

between $t_0 = (k-1)\Delta$ and $t_f = k\Delta$ and set $F_k := F(t_f)$. Then,

$$\Psi_A(0) = \Phi_A\left(N\Delta, (N-1)\Delta\right) \cdots \Phi_A\left(2\Delta, \Delta\right) \Phi_A\left(\Delta, 0\right) = F_N \cdots F_2 F_1$$

Expected advantage: Improved accuracy of computed eigenvalues using the periodic real Schur form of the N-periodic matrix F_k.

Computational example (cont.)

Results:

N	$\lvert\overline{\mu}_1\rvert$	$\overline{\mu}_2$
1	$3.9 \cdot 10^{-7}$	-2.05...
2	$2.6 \cdot 10^{-7}$	complex !!
5	$4.5 \cdot 10^{-7}$	-11.2653
10	$2.1 \cdot 10^{-9}$	-19.7407
25	$7.7 \cdot 10^{-12}$	-23.9921
50	$2.4 \cdot 10^{-15}$	-23.9982
100	$2.7 \cdot 10^{-15}$	-23.99995
200	$3.2 \cdot 10^{-15}$	-23.9999993
500	$1.9 \cdot 10^{-14}$	-23.999999998

10-digits accuracy possible by exploiting structure! $\Rightarrow$ similar approach possible to solve continuous-time periodic Riccati equations!

Multiple shooting method

Basic approach:

1. For $\Delta = T/N$, compute $X(0)$, $X(\Delta)$, ..., $X((N-1)\Delta)$, such that $X(t)$ satisfies $X(0) = X(T)$.
2. Integrate the appropriate differential equation to compute the solution on $[(k-1)\Delta, k\Delta]$, for $k = 1, \ldots, N$.

Computational aspects:

- computation of transition matrices over $[(k-1)\Delta, k\Delta]$, for $k = 1, \ldots, N$
- solution of appropriate periodic discrete-time Riccati, Lyapunov or Sylvester equations for which reliable algorithms are available.

Expected advantages:

- the effects of unstable dynamics significantly reduced for small Δ
- enhanced accuracy for large T

Periodic real Schur form (PRSF)

Fact 1: For given N-periodic F_k there exists orthogonal N-periodic Q_k s.t.

$$\widetilde{F}_k = Q_{k+1}^T F_k Q_k, \quad k = 1, \ldots, N$$

is in PRSF ($\widetilde{F}_1$ – quasi-upper triangular, $\widetilde{F}_2$, ..., $\widetilde{F}_N$ upper triangular)
$\Rightarrow Q_1^T \Psi_A(0) Q_1 = \widetilde{F}_N \cdots \widetilde{F}_2 \widetilde{F}_1$ is in a real Schur form (RSF)!

Fact 2: For given N-periodic F_k there exists orthogonal N-periodic Z_k s.t.

$$\widehat{F}_k = Z_{k+1}^T F_k Z_k = \begin{bmatrix} J_{k;11} & J_{k;12} \\ O & J_{k;22} \end{bmatrix}, \quad k = 1, \ldots, N$$

where $\Lambda(J_{N;11} \cdots J_{2;11} J_{1;11}) \in C_g$, $\Lambda(J_{N;22} \cdots J_{2;22} J_{1;22}) \in C \setminus C_g$
$\Rightarrow Z_1^T \Psi_A(0) Z_1 = \widehat{F}_N \cdots \widehat{F}_2 \widehat{F}_1$ in an ordered RSF.

Multiple shooting method (cont.)

1. For $\Delta = T/N$, compute $F_k := \Phi_A(k\Delta, (k-1)\Delta)$ for $k = 1, \ldots, N$.
2. Compute N-periodic orthogonal $Z_k = \begin{bmatrix} Z_{k;11} & Z_{k;12} \\ Z_{k;21} & Z_{k;22} \end{bmatrix}$ to reduce F_k to an ordered PRSF (i.e., $\Psi_H(0) = F_N \cdots F_2 F_1$ to an ordered RSF) s.t.

$$Z_{k+1}^T F_k Z_k = \begin{bmatrix} J_{k;11} & J_{k;12} \\ O & J_{k;22} \end{bmatrix},$$

   where $\Lambda(J_{N;11} \cdots J_{2;11} J_{1;11}) \in C_g$, $\Lambda(J_{N;22} \cdots J_{2;22} J_{1;22}) \in C \setminus C_g$.
3. Compute $X((k-1)\Delta) := Z_{k;21} Z_{k;11}^{-1}$ for $k = 1, \ldots, N$.

Interpretation of the solution

Denote $X_k := X((k-1)\Delta)$, define $P_k = \begin{bmatrix} I & 0 \\ X_k & I \end{bmatrix}$ and partition

$$F_k = \Phi_A(k\Delta, (k-1)\Delta) = \begin{bmatrix} F_{k,11} & F_{k,12} \\ F_{k,21} & F_{k,22} \end{bmatrix}$$

Since

$$\Phi_{\widetilde{A}}(k\Delta, (k-1)\Delta) = \begin{bmatrix} I & 0 \\ -X_{k+1} & I \end{bmatrix} \begin{bmatrix} F_{k,11} & F_{k,12} \\ F_{k,21} & F_{k,22} \end{bmatrix} \begin{bmatrix} I & 0 \\ X_k & I \end{bmatrix}$$

$$\Rightarrow -X_{k+1}F_{k,11} + F_{k,22}X_k - X_{k+1}F_{k,12}X_k + F_{k,21} = 0$$

– *non-standard discrete-time periodic Riccati equation*

Symmetric periodic Riccati differential equation

Main structural feature: $A(t)$ is Hamiltonian $\Rightarrow F_k := \Phi_A(k\Delta, (k-1)\Delta)$ is symplectic!

Step 1. Computation of F_k: employ numerical integrators able to guarantee that the resulting F_k is symplectic (e.g., symplectic Runge-Kutta methods) (Johansson *et al.*, 2007)

Step 2. Computation of the ordered PRSF of F_k: employ symplectic reduction (no algorithm presently available)

Step 3. Computation of solution $X(t)$ for $t \in [(k-1)\Delta, k\Delta]$: Special ODE solvers able to preserve the positivity of the numerical solution can be used (Dieci 1992, 1994).

Parallelization: Steps 1. and 3. are "embarrassingly" parallelizable!

Applications of multiple shooting technique

Periodic Lyapunov and Sylvester differential equations: efficient methods relying on discrete-time periodic Lyapunov and Sylvester equation solvers

Periodic system norms: $\|\mathcal{G}_{op}(\tau)\|$, where $\mathcal{G}_{op}(\tau)$ is the input-output operator from $u(t)$ to $y(t)$ for zero initial conditions $x(\tau) = 0$

$$y(t) = [\mathcal{G}_{op}(\tau)u](t), \; t \geq \tau$$

Multiple shooting techniques to compute Hankel-, $\mathcal{H}_2$- and $\mathcal{H}_\infty$-periodic system norms have been developed (Varga, 2005).

Output feedback stabilization of periodic systems:
evaluation of gradients of linear-quadratic criterion involves the solution of periodic Lyapunov differential equations

Some open computational problems

- solution of stabilization and pole assignment problems
- periodic output feedback stabilization
- efficient computation of frequency-responses (solution of linear system with block-Toeplitz + diagonal coefficient matrix)
- order reduction
- solution of various controller synthesis problems
- application of continuous matrix decompositions to solve various structural problems

Relevant contributions

Theory: Bittanti, Bolzern, Brocket, Colaneri, Hagiwara, Lampe, Rosenwasser, Zhou

Computational methods: Benner, Bojanczik, Byers, Granat, Guo, Hench, Higham, Johansson, Kagstrom, Kressner, Laub, Mehrmann, Rhee, Van Dooren, Varga

Geometric integration: Dieci, Hairer, Hench, Kenney, Laub, Lubich, MacLahan, Warnner